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NONLINEAR BENDING OF TOROIDAL SHELLS OF ARBITRARY TRANSVERSE CROSS SECTION LOADED WITH INTERNAL PRESSURE

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In this paper we derive complete geometrically nonlinear relations for the problem of the bending of a toroidal shell of arbitrary transverse cross section. An accurate expression is obtained for the potential of the internal-pressure forces, which holds for any distortions of the shape of the cross section. An algorithm and a numerical solution of the problems of the deformations of cylindrical and toroidal shells for large elastic displacements are considered. The results obtained are compared with existing analytic solutions and experimental data.

1. Introduction. Since the publication of the papers by Dubyaga [1] and Karman [2] there have been numerous investigations of the problem of the bending of thin-walled curvilinear tubes, most of which have been carried out using the linear theory of shells. The case of the combined action of internal pressure and bending moments on a tube of circular transverse cross section is considered in [3, 4] using variational principles. Another approach was employed in [5], which consists of solving the differential equations of the bending of a toroidal shell, first loaded with internal pressure. It was established that the stiffness properties and the stresses in the shell depend nonlinearly on the pressure. Small displacements were investigated and the problem was regarded as being linear in the bending moments. Large displacements for pure bending of cylindrical shells were considered in [6], and the value of the limiting moment for which a loss of the stability for the shells occurs was found, and the stability of a shell on bending, taking into account changes in its shape in the subcritical state was investigated for the first time. The results obtained in [6] were refined in [7-9] both by retaining small terms in the initial relations, and by choosing different approximating functions. The effect of the internal pressure when cylindrical shells are bent was taken into account in [10]. The previous results were generalized in [11] and two problems previously considered separately, were combined: the bending of curvilinear tubes in the linear formulation, and the deformation of cylindrical tubes in the case of large elastic displacements. The nonlinear equations of the bending of tubes with a small initial curvature of the axial line were derived and integrated approximately. The problem of the bending of curvilinear tubes loaded with an internal pressure was also solved in [12, 13], taking the geometrical nonlinearity into account.

It should be noted that the solutions mentioned above, particularly the nonlinear ones, were obtained using simplified deformation relations and retaining a small number of terms of the approximating series. It is therefore of interest to obtain more accurate results, particularly in the supercritical region.

Certain problems of the finite bending of curvilinear tubes were investigated in [14] using the nonlinear theory of shells.

2. Formulation of the Problem. We will regard the tube as a thin-walled toroidal shell. Suppose the tube is loaded with an internal pressure and boundary bending moments

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acting in the plane of its curvature. We will assume that the tube is fairly long so that the effect of the boundary conditions at the edges can be neglected.

Using the hypothesis of plane sections, we will write the equation of the middle surface of the shell before and after loading in the following vector form:

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r},\ \mathbf{r} = \mathbf{e}_i x_i,\ \mathbf{R}^{\vee} = \mathbf{R}_{0i}^{\vee} + \mathbf{r}^{\vee},\ \mathbf{r}^{\vee} = \mathbf{e}_i^{\vee} x_i^{\vee} \quad (i = 1, 2), \tag{2.1}$$

where $\mathbf{R}_0$ is the radius-vector of the axial line of the shell, $\mathbf{e}_i = \mathbf{e}_i(t)$ are the unit vectors in the plane of the transverse cross section, $x_i = x_i(s)$ are Cartesian coordinates of points in the transverse cross section, and t and s are the lengths of the arcs of the axial line and the contour of the transverse cross section. Here and henceforth we will use summation over repeated indices, and the superscript V denotes quantities which refer to the deformed state.

The problem of finding the deformed state of the shell reduces to deriving a parametric equation describing the form of the cross section $x_i^{\vee} = x_i^{\vee}(s)$. We will consider deformations of the middle surface of the shell that are small compared with unity. No limitations will be imposed on the displacements and the angles of rotation. Using (2.1) and the Kirchhoff–Love hypothesis $\varepsilon_s^z = \varepsilon_s + z\kappa_s$, $\varepsilon_t^z = \varepsilon_t + z\kappa_t$ (where z is the normal coordinate to the shell surface), we obtain the following equations for the deformations and the bendings:

$$\varepsilon_t = A_t^{-1}(\varepsilon + k^{\vee}x_1^{\vee} - kx_1),\ \varkappa_t = A_t^{-1}(k^{\vee}\lambda_1^{n\vee} - k\lambda_1^n),$$
$$\varepsilon_s = (1/2)(x_i^{\vee\cdot}x_i^{\vee\cdot} - 1),\ \varkappa_s = x_i^{\vee\cdot}\lambda_i^{n\vee\cdot} - x_i^{\cdot}\lambda_i^{n\cdot}$$

($A_t = 1 + kx_1$ is the Lamé parameter, ε and k are the deformation and curvature of the axial line, λ_i^n are the direction cosines of the normal to the shell surface, and the dot denotes a derivative with respect to the s coordinate).

The potential energy of deformation Π of a toroidal shell with unit length of the axial line has the form

$$\Pi = (1/2)\oint (T_t\varepsilon_t + M_t\varkappa_t + T_s\varepsilon_s + M_s\varkappa_s)A_t\,ds,$$

where T_t, T_s and M_t and M_s are the forces and bending moments connected with the deformations and curvatures of the middle surface by the relations of elasticity:

$$T_t = B(\varepsilon_t + \nu\varepsilon_s),\quad T_s = B(\varepsilon_s + \nu\varepsilon_t),$$
$$M_t = D(\varkappa_t + \nu\varkappa_s),\quad M_s = D(\varkappa_s + \nu\varkappa_t),\quad B = Eh/(1-\nu^2),\quad D = Bh^2/12$$

(E is Young's modulus, ν is Poisson's ratio, and h is the thickness of the shell).

The potential of the external forces acting on the shell can be written in the following form:

$$W = U - A,\ A = M(k^{\vee} - k)$$

(U is the potential of the forces of the internal pressure p, and A is the work of the external bending moments M).

The potential of the internal-pressure forces is taken with the opposite sign to the work of the internal-pressure forces for a change in the volume bounded by the shell. The complete expression for the potential has the form

$$U = -(1/2)\,p\left((1+\varepsilon)\oint (x_1^{\vee}x_2^{\vee\cdot} - x_1^{\vee\cdot}x_2^{\vee})\,ds + k\oint x_2^{\vee\cdot}x_1^{\vee 2}\,ds\right) + pV \tag{2.2}$$

(V_k is a constant, and V is the volume in the undeformed state).

3. An Algorithm for Solving the Problem. To determine the deformed state of the shell we will use the method of local approximation of the deformation relations for an element of the contour [15]. Using an expansion of the unknown functions in a Taylor series and

neglecting terms of order $O(\ell^2)$ for a small element of length ℓ, we obtain the following approximation relations:

$$\varepsilon_t = A_t^{-1}(\varepsilon + k^\vee x_1^\vee - kx_1),\ \varkappa_t = A_t^{-1}(k^\vee \lambda_1^{n\vee} - k\lambda_1^n), \tag{3.1}$$
$$\varepsilon_s = (1/2)(b_i b_k x_{ji}^\vee x_{jk}^\vee - 1),$$
$$\varkappa_s = N_i\theta_i,\ \theta_i = b_k(\lambda_{ji}^{n\vee} x_{jk}^\vee - \lambda_{ji}^n x_{jk}),$$
$$A_t = 1 + kx_1,\ x_1 = (1/2)(x_{11} + x_{12}),\ \lambda_1^n = (1/2)(\lambda_{11}^n + \lambda_{12}^n),$$
$$b_1 = -b_2 = -1/l,\ N_1 = (6s - 4l)/l^2,\ N_2 = (6s - 2l)/l^2.$$

Here x_{ji} and λ_{ji}^n are the nodal values of the coordinates and the direction cosines of the unit vector of the normal for a finite element.

The potential of the internal-pressure forces of the discrete system (2.2) can be written in the form of the sum of the contributions of each of the elements:

$$U = \sum_k U_k,\ U_k = -p\left[(1+\varepsilon)\omega + \frac{1}{6}k\left(2\omega(x_{11}^\vee + x_{12}^\vee) + x_{22}^\vee x_{12}^{\vee 2} - x_{11}^{\vee 2} x_{21}^\vee\right)\right] + pV_k,\ \ \omega = \frac{1}{2}(x_{11}^\vee x_{22}^\vee - x_{12}^\vee x_{21}^\vee)$$

The main problem when using the energy method is to calculate the first and second variations of the energy, which are necessary in order to formulate the conditions of equilibrium and for the iterative process of the solution. To obtain an algorithm for the calculations we will introduce two levels of discrete parameters, representing the vectors of the generalized elastic displacements $\mathbf{u}$ and the generalized coordinates $\mathbf{q}$ of an element:

$$\mathbf{u}^T = |\varepsilon_s,\ \theta_1,\ \theta_2,\ \varepsilon_t,\ \varkappa_t|, \tag{3.2}$$
$$\mathbf{q}^T = |x_{11}^\vee, x_{21}^\vee, \varphi_1^\vee, x_{12}^\vee, x_{22}^\vee, \varphi_2^\vee, \varepsilon, k^\vee|$$

($\phi_i^\vee$ is the angle of rotation of the normal at the i-th node).

Using (3.1) and (3.2) we can represent the potential energy of an element in the form

$$\Pi = (1/2)\mathbf{u}^T\mathbf{K}\mathbf{u}.$$

where $\mathbf{K}$ is a symmetric stiffness matrix with the following nonzero components:

$$K_{11} = BA_t l,\ K_{14} = \nu K_{11},\ K_{22} = D[4 + k(3x_{11} + x_{12})]/l,$$
$$K_{23} = 2DA_t/l,\ K_{25} = -\nu D(1 + kx_{11}),$$
$$K_{33} = D[4 + k(x_{11} + 3x_{12})]/l,\ \ K_{35} = \nu D(1 + kx_{12}),\ \ K_{44} = K_{11},\ \ K_{55} = DA_t l.$$

The condition for the total potential energy to be stationary $\delta(\Pi + W) = 0$ leads to the following system of nonlinear algebraic equations:

$$\mathbf{H}\delta\mathbf{q} + \mathbf{g} = 0. \tag{3.3}$$

Here $\mathbf{H}$ and $\mathbf{g}$ are the Hess matrix and the gradient, which can be calculated from the formulas

$$\mathbf{g} = \mathbf{u}'\mathbf{P} + \mathbf{g}^p,\ \mathbf{P} = \mathbf{K}\mathbf{u}, \tag{3.4}$$
$$\mathbf{H} = \mathbf{u}'\mathbf{K}(\mathbf{u}')^T + P_i\mathbf{u}_i'' + \mathbf{H}^p \quad (i = 1, \ldots, 5)$$

($\mathbf{u}'$ and $\mathbf{u}_i''$ are matrices of the first and second derivatives of the components of the vector $\mathbf{u}$ with respect to the components of the vector $\mathbf{q}$). The nonzero components of the matrix $\mathbf{u}'$ have the form

$$\partial\varepsilon_s/\partial x_{ij}^\vee = b_j b_k x_{ik}^\vee,\ \partial\theta_i/\partial x_{jk}^\vee = b_k\lambda_{ji}^{n\vee},$$
$$\partial\theta_i/\partial\varphi_i^\vee = b_k x_{jk}^\vee\lambda_{ji}^\vee,\ \partial\varepsilon_t/\partial x_{ij}^\vee = (1/2)k^\vee A_t^{-1},$$
$$\partial\varepsilon_t/\partial\varepsilon = A_t^{-1},\ \partial\varepsilon_t/\partial k^\vee = x_1^\vee A_t^{-1},$$
$$\partial\varkappa_t/\partial\varphi_i^\vee = (1/2)k^\vee\lambda_{1i}^\vee A_t^{-1},\ \partial\varkappa_t/\partial k^\vee = \lambda_1^{n\vee}A_t^{-1}$$

(λ^{V}_{ji} are the direction cosines of the unit vector, tangential to the deformed contour of the transverse cross section at the i-th node). The nonzero components of the matrices of the second derivatives are given by the relations

$$\partial^2\varepsilon_s/\partial x^{V}_{ij}\partial x^{V}_{ih} = b_j b_k,\ \partial^2\theta_i/\partial x^{V}_{jk}\partial\varphi^{V}_i = b_k\lambda^{V}_{ji},$$
$$\partial^2\theta_i/\partial\varphi_i^{V2} = -\,b_k x^{V}_{jk}\lambda^{nV}_{ji},\quad \partial^2\varepsilon_t/\partial x^{V}_{1j}\partial k^{V} = (1/2)\,A_t^{-1},$$
$$\partial^2\varkappa_t/\partial\varphi^{V}_i\partial k^{V} = (1/2)\,\lambda^{V}_{1i}A_t^{-1},\ \partial^2\varkappa_t/\partial\varphi_i^{V2} = -\,(1/2)\,k^{V}\lambda^{nV}_{1i}A_t^{-1}.$$

The vector $\mathbf{g}^p$ and the matrix $\mathbf{H}^p$ which occur in expression (3.4) reflect the effect of the internal pressure and have the following nonzero components:

$$g_1^p = -\frac{1}{2}\,p\left[(1+\varepsilon)\,x^{V}_{22} + \frac{1}{3}\,k^{V}a_1a_2\right],\ g_2^p = \frac{1}{2}\,p\left[(1+\varepsilon)\,x^{V}_{12} + \frac{1}{3}\,k^{V}a_3\right],$$
$$g_4^p = \frac{1}{2}\,p\left[(1+\varepsilon)\,x^{V}_{21} - \frac{1}{3}\,k^{V}a_1a_4\right],\ g_5^p = -\frac{1}{2}\,p\left[(1+\varepsilon)\,x^{V}_{11} + \frac{1}{3}\,k^{V}a_3\right],$$
$$g_7^p = -\,p\omega,\ g_8^p = -\frac{1}{6}\,pa_1a_3,\ h_{11}^p = -\frac{1}{3}\,pk^{V}a_1,$$
$$h_{12}^p = \frac{1}{6}\,pk^{V}a_2,\ h_{14}^p = -\frac{1}{6}\,pk^{V}a_1,\ h_{15}^p = -\frac{1}{2}\,p\left[1+\varepsilon+\frac{1}{3}\,k^{V}a_2\right],$$
$$h_{17}^p = -\frac{1}{2}\,px^{V}_{22},\ h_{18}^p = -\frac{1}{6}\,pa_1a_2,\ h_{24}^p = \frac{1}{2}\,p\left(1+\varepsilon+\frac{1}{3}\,k^{V}a_1\right),$$
$$h_{27}^p = \frac{1}{2}\,px^{V}_{12},\ h_{28}^p = \frac{1}{6}\,pa_3,\ h_{44}^p = -\frac{1}{3}\,pk^{V}a_1,\ h_{45}^p = -\frac{1}{6}\,pk^{V}a_4,$$
$$h_{47}^p = \frac{1}{2}\,px^{V}_{21},\ h_{48}^p = -\frac{1}{6}\,pa_1a_2,\ h_{57}^p = -\frac{1}{2}\,px^{V}_{11},\ h_{58}^p = -\frac{1}{6}\,pa_3,$$
$$a_1 = x^{V}_{22} - x^{V}_{21},\ a_2 = 2x^{V}_{11} - x^{V}_{12},\ a_3 = x_{11}^{V2} + x^{V}_{11}x^{V}_{12} + x_{12}^{V2},\ a_4 = x^{V}_{11} + 2x^{V}_{12}.$$

Since $\mathbf{g}$ and $\mathbf{H}$ depend on $\mathbf{q}$, the process of the solution using scheme (3.3) is iterative and is continued until the conditions of equilibrium are satisfied with a specified degree of accuracy. We choose the vector $\mathbf{q}$ corresponding to the undeformed state of the shell as the initial approximation. A linear solution of the problem is obtained after a single iteration.

The algorithm described can be used to calculate cylindrical and toroidal shells with arbitrary transverse cross section. One needs as initial data a knowledge of the values of the coordinates and the direction cosines of the normal at the nodes of the contour of the transverse cross section.

4. Results of the Calculations. We will consider, as an application of the above algorithm, the problem of the bending of curvilinear tubes with a radius of the transverse cross section r, loaded in advance with an internal pressure. In Fig. 1 the numerical results obtained for the flexibility f are compared with experimental data from [3]. Curve 1 is for a tube with the parameters ak = 0.15 and a/h = 93.84, while curve 2 is for a tube with parameters ak = 0.11 and a/h = 93.84. It can be seen that the results of the numerical solution agree well with the experimental data.

In Figs. 2 and 3 we show the results of nonlinear deformation and the stability of toroidal shells of circular cross section with initial curvature parameter $\mu = 5$, where $\mu = (12(1 - \nu^2))^{1/2} \times r^2k/h$, $\nu = 0.3$. The continuous curves represent the dimensionless moment $m = (12(1 - \nu^2))^{1/2}Mr^2/hEI$ as a function of the bending parameter of the axial line $\alpha = (12(1 - \nu^2))^{1/2}r^2(k^{V} - k)/h$, for different levels of pressure, characterized by the quantity $\bar{p} = 4(1 - \nu^2)(r/h)^3p/E$ ($I = \pi r^3h$ is the moment of inertia of the transverse cross section). The results obtained are compared with the solution given in [11], which is represented by the dashed curves. In the region of positive values of the bending parameter the formula from [11] gives lower values of the critical parameters α_c and m_c, while in the region of negative values of the bending parameter it gives higher values of α_c and m_c. For $\alpha > 0$ the error in determining the critical moment reaches 38%, while the error in determining the critical bending reaches 50%; for $\alpha < 0$, the errors amount to 7% and 15% respectively.

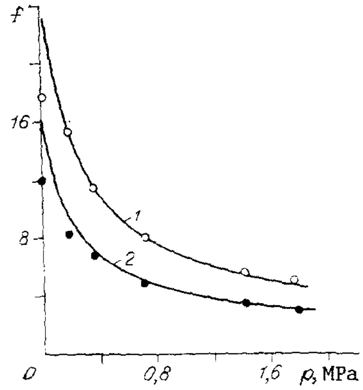

Fig. 1

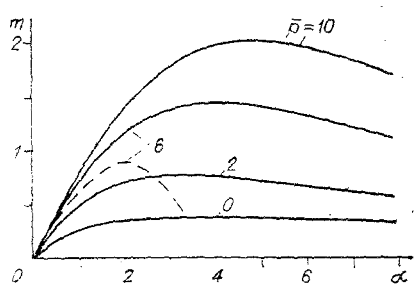

Fig. 2

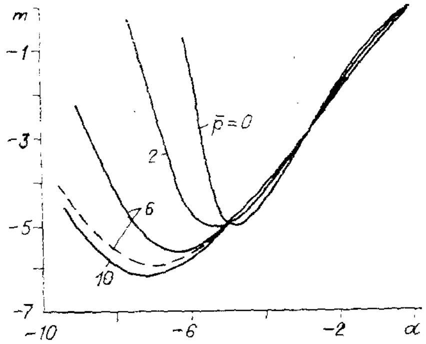

Fig. 3

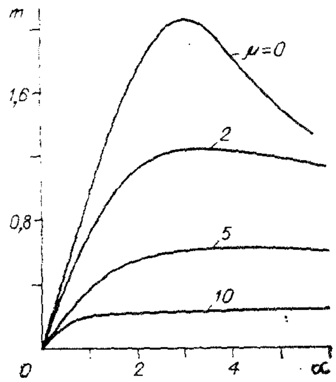

Fig. 4

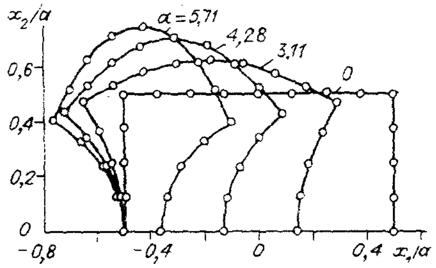

Fig. 5

TABLE 1

μ	N				
	8	12	16	20	24
	m_c				
0	1,089	1,072	1,065	1,063	1,062
10	0,275	0,290	0,295	0,298	0,299
50	0,104	0,128	0,138	0,143	0,145

Note that for unbending of a shell, when there is no pressure ($\bar{p} = 0$), a loss in stability occurs when $m_c = \alpha_c = -\mu$. This result confirms the conclusions reached in [13].

It follows from the above analysis that the solutions of the problem of pure bending of tubes obtained previously [10-12], satisfactorily describe the deformation only at the initial stage of the loading, where the deviation of the transverse cross section from a circular form is small. In the supercritical region these solutions become inadmissible, which can be explained by the simplified relations and limited number of terms of the approximating series employed.

The algorithm developed above can also be applied to the bending of shells which have transverse cross sections with corner points. The case of the bending of a curvilinear tube with a rectangular transverse cross section is described in the linear formulation in [16]. The nonlinear problem has not so far been considered.

In Fig. 4 we show the nonlinear characteristics for a square transverse cross section of side a. The parameters of the load, the bending, and the initial curvature of the axial line are defined by the formulas

$$m = (3(1-\nu^2))^{1/2} Ma^2/hEI, \alpha = (3(1-\nu^2))^{1/2}(k^V - k)a^2/h,$$
$$\mu = (3(1-\nu^2))^{1/4}(k/h)^{1/2}a$$

($I = (2/3)a^3h$ is the moment of inertia of the cross section). A characteristic feature is the fact that degeneration of the limiting point occurs as the initial curvature increases ($\mu \gtrsim 10$). In Figure 5 we show on a real scale, the forms of the deformed transverse cross section for different values of the bending parameter for $\mu = 0$.

In conclusion we present the results of an investigation of the convergence of the above algorithm. Table 1 shows values of the critical parameters m_c for toroidal shells with a circular transverse cross section as a function of the number of elements N which fit into half the contour of the section.

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